Lecture 12 (1/31/22)

Recall.

- A *meromorphic* function f in $G \subseteq \mathbb{C}$ is analytic with poles in G.
  - If we define f to be ∞ at each pole, then $f : G \to \mathbb{C}_\infty$ is a continuous function.

Thus, we may consider: $\mathcal{M}(G) \subseteq C(G, \mathbb{C}_\infty)$.

- $\mathcal{M}(G)$ - the space of continuous functions with metric induced by $(C(G, \mathbb{C}_\infty), \rho)$, where $d = d_\infty$ is Fubini-Study metric on $\Omega = \mathbb{C}_\infty$.

- $\mathbb{C} = \mathbb{C}_\infty \setminus \{\infty\}$. Metrics on almost comparable: (i) $B(z,r) \sim B_\infty(z,r)$ for $z \in \mathbb{C}$. (ii) $B_\infty(\infty, \rho) \sim \mathbb{C}_\infty \setminus B(0, \frac{1}{\rho})$.
  $\Rightarrow$ $\{z_n\}_{n=1}^\infty$ in $\mathbb{C}$ converges to
  (i') $z \in \mathbb{C}$ if either $|z - z_n| \to 0$ or $d_\infty(z, z_n) \to 0$.
  (ii') $\infty \in \mathbb{C}_\infty \Leftrightarrow |z_n| \to \infty$.

Thm1. Let $\{f_n\}_{n=1}^{\infty}$ be seq. in $\mathcal{M}(G)$ (region)
s.t. $f_n \to f$ in $\mathcal{C}(G, \mathbb{C}_\infty)$. Then, either
$f \equiv \infty$ or $f \in \mathcal{M}(G)$. If the f_n are
analytic (no poles), then either $f \equiv \infty$ or
analytic.

Pf. (1) If $a \in G$ s.t. $f(a) \neq \infty \Rightarrow \exists B_\infty = B_\infty(0, \rho)$ ($\infty \notin \overline{B_\infty}$)
s.t. $f(a) \in B_\infty$. $\Rightarrow G' = f^{-1}(B_\infty)$ open
$\subseteq G$, $a \in G'$. Let $B(a,\delta)$ be s.t. $\overline{B(a,\delta)}$
$\subset\subset G'$. Since $f_n \to f$ unif. on $K = \overline{B(a,\delta)}$
wrt d_∞ $\exists N$ s.t. $f_n(K) \subseteq B_\infty$ and
hence f_n analytic in $B(a,\delta)$ for $n \geq N$.
By (i'), $f_n \to f$ in K wrt $|\cdot|$ on $\mathbb{C}$
$\Rightarrow f$ is analytic on $B(a,\delta)$.

Next, since $w \to 1/w$ is an isometry
on $\mathbb{C}_\infty$, $\frac{1}{f_n} \to \frac{1}{f}$ in $\mathcal{C}(G, \mathbb{C}_\infty)$
as well. (2) If $a \in G$ s.t. $f(a) = \infty$,
the argument above shows that

$\tilde{f} = \frac{1}{f}$ is analytic in $B(a,\delta)$ and the analytic $\tilde{f}_n = \frac{1}{f_n} \to \tilde{f}$ unif. in K wrt $|\cdot|$ on $\mathbb{C}$. By Hurwitz, either $\tilde{f} \equiv 0$ in $B(a,\delta)$ or $\tilde{f}$ has only finite number $m \geq 1$ of zeros in $B(a,\delta)$ in which case, for $n >> 1$, each $\tilde{f}_n$ also has m zeros in $B(a,\delta)$.

To summarize, each $a \in G$ either has a ball $B(a,\delta)$ of type ① or ②. A $B(a,\delta)$ of type ② is either such that $f \equiv \infty$ in $B(a,\delta)$ or f is meromorphic with $m \geq 1$ poles. In the latter case, the f_n must also have m poles, $n >> 1$.

If $\exists\, B(a,\delta)$ where $f \equiv \infty$, then
$$G' = \text{int } f^{-1}(\infty) = \{a : \exists\, B(a,\delta) \text{ where } f \equiv \infty\}$$
is not empty. G' is open by construction.

If $a \in \overline{G'}$, then $f(a) = \infty$. There is a type ② $B(a,\delta)$ s.t. either f has finite # poles or $f \equiv \infty$ in $B(a,\delta)$. Since a is a limit point of G', only the second case is possible. $\Rightarrow a \in G' \Rightarrow G' = G$ by connectedness.

Thus, if $f \not\equiv \infty$, then f is meromorphic in G and if it has poles, then for $n \gg 1$, the f_n have poles (near poles of f). This completes pf. $\boxtimes$

<u>Cor</u> $M(G) \cup \{\infty\}$ is complete.